VECTOR SPACES

BRANKO ĆURGUS

Contents

In these notes we denote $\mathbb{C}$ denotes the set of all complex numbers, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{Z}$ denotes the set of all integers and $\mathbb{N}$ denotes the set of all positive integers.

1. Axioms

Definition 1.1. In these notes $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. Since both $\mathbb{R}$ or $\mathbb{C}$ are fields, we will sometimes refer to $\mathbb{F}$ as a field of scalars. $\lhd$

Definition 1.2. Let $\mathscr{V}$ be a nonempty set. The set $\mathscr{V}$ is called a *vector space over* $\mathbb{F}$ if the following ten axioms are satisfied.

- **AE.** There exists a function $+ : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$, called *addition* in $\mathscr{V}$. Its value at a pair $(u, v) \in \mathscr{V} \times \mathscr{V}$ is denoted by $u + v$.
- **AA.** For all $u, v, w \in \mathscr{V}$ we have $u + (v + w) = (u + v) + w$.
- **AC.** For all $u, v \in \mathscr{V}$ we have $u + v = v + u$.
- **AZ.** There exists an element $0_{\mathscr{V}} \in \mathscr{V}$ such that $v + 0_{\mathscr{V}} = v$ for all $v \in \mathscr{V}$.
- **AO.** For each $v \in \mathscr{V}$ there exists $w \in \mathscr{V}$ such that $v + w = 0_{\mathscr{V}}$.
- **SE.** There exists a function $\cdot : \mathbb{F} \times \mathscr{V} \to \mathscr{V}$, called *scaling* in $\mathscr{V}$. Its value at a pair $(\alpha, v) \in \mathbb{F} \times \mathscr{V}$ is denoted by $\alpha \cdot v$, or simply αv.
- **SA.** For all $\alpha, \beta \in \mathbb{F}$ and all $v \in \mathscr{V}$ we have $\alpha(\beta v) = (\alpha\beta)v$.
- **SDV.** For all $\alpha \in \mathbb{F}$ and all $u, v \in \mathscr{V}$ we have $\alpha(u + v) = \alpha u + \alpha v$.
- **SDS.** For all $\alpha, \beta \in \mathbb{F}$ and all $v \in \mathscr{V}$ we have $(\alpha + \beta)v = \alpha v + \beta v$.
- **SO.** For all $v \in \mathscr{V}$ we have $1v = v$.

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Explanation of abbreviations. AE - Addition exists; AA - Addition is associative; AC - Addition is commutative; AZ - Addition has a zero vector (additive identity); AO - Addition has opposites (additive inverses); SE - Scalar multiplication exists (scaling exists); SA - Scalar multiplication is associative with field multiplication; SDV - Scalar multiplication is distributive over vector addition; SDS - Scalar multiplication is distributive over scalar addition; SO - Scalar multiplication with the number one is special.

Remark 1.3. In Axioms SA and SDS I used different colors to emphasize that two different binary operations appear in the same equality. In Axiom SA, yellow ▮ marks scalar multiplication in $\mathscr{V}$, while light blue ▮ marks multiplication in the field $\mathbb{F}$. In Axiom SDS, the yellow plus $+$ marks addition in $\mathscr{V}$, while the light blue plus $+$ marks addition in $\mathbb{F}$. I use this color convention only in these two axioms; the reader is expected to internalize the distinction wherever it occurs. See Problem 7.1 for a more detailed discussion of this point. ◁

2. Basic propositions

A few immediate consequences of Definition 1.2 are presented in the following propositions.

Proposition 2.1. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$. The zero vector $0_{\mathscr{V}}$ from axiom* **AZ** *is unique. That is, if $w \in \mathscr{V}$ has the property that for all $v \in \mathscr{V}$ we have $v + w = v$, then $w = 0_{\mathscr{V}}$.*

Proof. Let $w \in \mathscr{V}$ be such that for all $v \in \mathscr{V}$ we have $v + w = v$. Using axiom **AZ** with $v = w$, we have $w + 0_{\mathscr{V}} = w$. Using the assumed property with $v = 0_{\mathscr{V}}$, we have $0_{\mathscr{V}} + w = 0_{\mathscr{V}}$. By commutativity of addition (axiom **AC**), $w + 0_{\mathscr{V}} = 0_{\mathscr{V}} + w$, hence

$$w = w + 0_{\mathscr{V}} = 0_{\mathscr{V}} + w = 0_{\mathscr{V}}.$$

Therefore $w = 0_{\mathscr{V}}$. □

Proposition 2.2. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$. For every $\alpha \in \mathbb{F}$ and every $v \in \mathscr{V}$ the following equivalence holds:*

$$\alpha v = 0_{\mathscr{V}} \quad \Leftrightarrow \quad \alpha = 0 \ \vee \ v = 0_{\mathscr{V}}. \tag{2.1}$$

Proof. First we prove the "if" part ($\Leftarrow$) in (2.1). The proof is in two parts. Let $v \in \mathscr{V}$ be arbitrary and let $\alpha = 0$. Then by **SE** we have that $0v \in \mathscr{V}$.

By **AO** there exists $w \in \mathscr{V}$ such that $0v + w = 0_{\mathscr{V}}$. Then

$$\begin{aligned} 0_{\mathscr{V}} &= 0v + w && \boxed{\text{by the choice of } w} \\ &= (0+0)v + w && \boxed{\text{since } 0+0=0 \text{ in } \mathbb{C}, \textbf{SE} \text{ and } \textbf{AE}} \\ &= (0v+0v) + w && \boxed{\text{by } \textbf{SD}, \textbf{AE}} \\ &= 0v + (0v + w) && \boxed{\text{by } \textbf{AA}} \\ &= 0v + 0_{\mathscr{V}} && \boxed{\text{by the choice of } w, 0v + w = 0_{\mathscr{V}}, \textbf{AE}} \\ &= 0v && \boxed{\text{by } \textbf{AZ}}. \end{aligned}$$

This sequence of equalities proves $0v = 0_{\mathscr{V}}$.

Let $v = 0_{\mathscr{V}}$ and let $\alpha \in \mathbb{F}$ be arbitrary. Then by **SE** we have that $\alpha 0_{\mathscr{V}} \in \mathscr{V}$. By **AO** there exists $w \in \mathscr{V}$ such that $\alpha 0_{\mathscr{V}} + w = 0_{\mathscr{V}}$. Then

$$\begin{aligned} 0_{\mathscr{V}} &= \alpha 0_{\mathscr{V}} + w && \boxed{\text{by the choice of } w} \\ &= \alpha\bigl(0_{\mathscr{V}} + 0_{\mathscr{V}}\bigr) + w && \boxed{\text{by } \textbf{AZ}, \textbf{SE} \text{ and } \textbf{AE}} \\ &= \bigl(\alpha 0_{\mathscr{V}} + \alpha 0_{\mathscr{V}}\bigr) + w && \boxed{\text{by } \textbf{SD}, \text{ and } \textbf{AE}} \\ &= \alpha 0_{\mathscr{V}} + \bigl(\alpha 0_{\mathscr{V}} + w\bigr) && \boxed{\text{by } \textbf{AA}} \\ &= \alpha 0_{\mathscr{V}} + 0_{\mathscr{V}} && \boxed{\text{by the choice of } w \text{ and } \textbf{AE}} \\ &= \alpha 0_{\mathscr{V}} && \boxed{\text{by } \textbf{AZ}}. \end{aligned}$$

This sequence of equalities proves $\alpha 0_{\mathscr{V}} = 0_{\mathscr{V}}$. This completes the "if" part ($\Leftarrow$) in (2.1).

Now we prove the "only if" part ($\Rightarrow$) in (2.1). This implication is of the form $p \Rightarrow q \vee r$, where p, q, r are mathematical statements. The implication $p \Rightarrow q \vee r$ is equivalent to the implication $p \wedge \neg q \Rightarrow r$, since the negations of these implications are identical. We proceed to prove

$$\alpha v = 0_{\mathscr{V}} \ \wedge \ \alpha \neq 0 \quad \Rightarrow \quad v = 0_{\mathscr{V}}. \tag{2.2}$$

Let $\alpha \in \mathbb{F}$ and $v \in \mathscr{V}$ be arbitrary and assume $\alpha v = 0_{\mathscr{V}}$ and $\alpha \neq 0$. Since $\alpha \in \mathbb{F} \setminus \{0\}$, we have that $1/\alpha \in \mathbb{F}$. Then

$$\begin{aligned} 0_{\mathscr{V}} &= \bigl(1/\alpha\bigr)0_{\mathscr{V}} && \boxed{\text{by } \Leftarrow \text{ in (2.1)}} \\ &= \bigl(1/\alpha\bigr)\bigl(\alpha v\bigr) && \boxed{\text{by } \textbf{SE} \text{ and } \alpha v = 0_{\mathscr{V}}} \\ &= \bigl((1/\alpha)\alpha\bigr)v && \boxed{\text{by } \textbf{SA}} \\ &= 1v && \boxed{\text{by } \textbf{SE} \text{ and } (1/\alpha)\alpha = 1 \text{ in } \mathbb{F}} \\ &= v && \boxed{\text{by } \textbf{SO}}. \end{aligned}$$

This sequence of equalities proves (2.2). Since (2.2) is equivalent to the "only if" part ($\Rightarrow$) in (2.1), the proposition is proved. $\square$

Proposition 2.3. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$. For every $v \in \mathscr{V}$ the following equivalence holds*

$$v + w = 0_{\mathscr{V}} \quad \Leftrightarrow \quad w = (-1)v. \tag{2.3}$$

Proof. Let $v \in \mathscr{V}$ be arbitrary. First we will prove the "if" part ($\Leftarrow$) in (2.3). Let $w = (-1)v$. Then

$$\begin{aligned} v + w &= v + (-1)v && \boxed{\text{by } \mathbf{AE}} \\ &= 1v + (-1)v && \boxed{\text{by } \mathbf{SO}\text{, and } \mathbf{AE}} \\ &= \bigl(1 + (-1)\bigr)v && \boxed{\text{by } \mathbf{SD}} \\ &= 0v && \boxed{\text{by } 1 + (-1) = 0 \text{ in } \mathbb{C} \text{ and } \mathbf{SE}} \\ &= 0_{\mathscr{V}} && \boxed{\text{by Proposition 2.2}} \end{aligned}$$

The presented sequence of equalities proves the "if" part ($\Leftarrow$) in (2.3).

Next we prove the converse, that is we prove the "only if" part ($\Rightarrow$) in (2.3). Assume $v + w = 0_{\mathscr{V}}$. Then

$$\begin{aligned} w &= 0_{\mathscr{V}} + w && \boxed{\text{by } \mathbf{AZ} \text{ and } \mathbf{AC}} \\ &= 0v + w && \boxed{\text{by Proposition 2.2 and } \mathbf{AE}} \\ &= \bigl((-1) + 1\bigr)v + w && \boxed{\text{by } (-1) + 1 = 0 \text{ in } \mathbb{C}, \mathbf{SE} \text{ and } \mathbf{AE}} \\ &= \bigl((-1)v + 1v\bigr) + w && \boxed{\text{by } \mathbf{SD} \text{ and } \mathbf{AE}} \\ &= (-1)v + \bigl(v + w\bigr) && \boxed{\text{by } \mathbf{AA}, \mathbf{SO} \text{ and } \mathbf{AE}} \\ &= (-1)v + 0_{\mathscr{V}} && \boxed{\text{by } v + w = 0_{\mathscr{V}}\text{,and } \mathbf{AE}} \\ &= (-1)v && \boxed{\text{by } \mathbf{AZ}} \end{aligned}$$

The presented sequence of equalities proves the "only if" part ($\Rightarrow$) in (2.3).

Since $v \in \mathscr{V}$ was arbitrary, the proposition is proved. $\square$

Definition 2.4. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $v \in \mathscr{V}$. The unique solution of equation $v + x = 0_{\mathscr{V}}$ is denoted by $-v$ and it is called the *opposite* of v. For $u, v \in \mathscr{V}$ instead of $u + (-v)$ we write $u - v$. $\triangleleft$

Definition 2.5. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$, let $v_k \in \mathscr{V}$ for every $k \in \mathbb{N}$, and let $n \in \mathbb{N}$. The sum

$$\sum_{k=1}^{n} v_k$$

is defined as follows: If $n = 1$ set

$$\sum_{k=1}^{1} v_k = v_1.$$

If $n \in \mathbb{N} \setminus \{1\}$ we use the definition by the finite recursion:

$$\forall m \in \{2, \ldots, n\} \quad \text{we set} \quad \sum_{k=1}^{m} v_k = \left(\sum_{k=1}^{m-1} v_k \right) + v_m \qquad \triangleleft$$

For example, if $v_1, v_2, v_3, v_4, v_5 \in \mathscr{V}$, then

$$v_1 + v_2 + v_3 + v_4 + v_5 = \Big(\big((v_1 + v_2) + v_3 \big) + v_4 \Big) + v_5$$

Definition 2.6. Let $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, and $v_1, \ldots, v_n \in \mathscr{V}$. The expression

$$\sum_{k=1}^{n} \alpha_k v_k = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

is called a *linear combination* of the vectors $v_1, \ldots, v_n$ in $\mathscr{V}$. A linear combination is said to be *trivial* if $\alpha_1 = \cdots = \alpha_n = 0$; otherwise, it is called *nontrivial*. $\triangleleft$

3. Examples

Example 3.1. The simplest vector space is the vector space which consists of only one element, which must be the zero vector, denote it by $\mathbf{0}$. (I use the bold-face zero to make it different from the real number 9. Since it is the only element in this space we could denote it whatever we want.) Thus $\{\mathbf{0}\}$, the set which consists of only one element is a vector space over $\mathbb{F}$. This vector space is called the *zero vector space* or *trivial vector space*. The definition of adition is $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and the definition of the scalar multiplication is $\alpha \mathbf{0} = \mathbf{0}$ for all $\alpha \in \mathbb{F}$.

One really needs to be careful with the zero vector space. Sometimes I think of it as a scary vector space. Many propositions hold for all vector spaces except for the zero vector space. When writing such propositions, it is easy to forget to exclude it and end up with a technically incorrect statement. $\triangleleft$

Example 3.2. Setting $\mathscr{V} = \mathbb{F}$, then $\mathscr{V}$ is a vector space over $\mathbb{F}$. The addition in $\mathscr{V} = \mathbb{F}$ is the addition of complex numbers in $\mathbb{F}$ and the scaling in $\mathscr{V} = \mathbb{F}$ is just the multiplication of complex numbers. The axioms of the vector space then follow from the axioms of the axioms of real numbers if $\mathbb{F} = \mathbb{R}$ or axioms of the complex numbers if $\mathbb{F} = \mathbb{C}$. $\triangleleft$

Example 3.3. This is the quintessential example of a vector space. Many other specific vector spaces are special cases of this example. Let D be an arbitrary nonempty set. Let $\mathscr{V}$ be the set of all functions from D to $\mathbb{F}$. This set is denoted by $\mathbb{F}^D$. The addition in $\mathbb{F}^D$ is defined as follows: let $f, g \in \mathbb{F}^D$, the function $f + g$ is defined by

$$(f + g)(t) := f(t) + g(t) \quad \text{for all} \quad t \in D.$$

The scaling in $\mathbb{F}^D$ is defined as follows: let $\alpha \in \mathbb{F}$ and $f \in \mathbb{F}^D$, the function αf is defined by

$$(\alpha f)(t) := \alpha f(t) \quad \text{for all} \quad t \in D.$$

The above definitions of addition and scaling of functions are called *pointwise* definitions. As an exercise you should go through the proofs of all the axioms of the vector space for this specific case.

It is important to highlight some prominent functions in $\mathbb{F}^D$. The first among them are the *constant functions*. For an arbitrary fixed $c \in \mathbb{F}$, define $f(t) = c$ for all $t \in D$.

The second are the *indicator functions*. For an arbitrary subset $A \subseteq D$, define

$$\chi_A(t) = \begin{cases} 1 & \text{if} \quad t \in A, \\ 0 & \text{if} \quad t \in D \setminus A. \end{cases}$$

In particular, for an arbitrary fixed $s \in D$ and the singleton set $A = \{s\}$, we have

$$\chi_{\{s\}}(t) = \begin{cases} 1 & \text{if} \quad t = s, \\ 0 & \text{if} \quad t \in D \setminus \{s\}. \end{cases}$$

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Example 3.4. This is a special case of Example 3.3. Let $n \in \mathbb{N}$ and define

$$D = \{t \in \mathbb{N} : t \leq n\}.$$

This set is often written simply as $D = \{1, \ldots, n\}$. The vector space $\mathbb{F}^D$ can be naturally identified with $\mathbb{F}^n$, the space of all n-tuples of elements of $\mathbb{F}$.

Specifically, we identify the n-tuple $(v_1, \ldots, v_n) \in \mathbb{F}^n$ with the function $f \in \mathbb{F}^D$ defined by

$$f(k) = v_k \quad \text{for all} \quad k \in \{1, \ldots, n\}.$$

For

$$\mathbf{x} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \in \mathbb{F}^D,$$

this notation reads as follows:

$$\mathbf{x}(k) = \xi_k \quad \text{for all} \quad k \in \{1, \ldots, n\}.$$

At first glance, this approach may seem somewhat obscure, but it is commonly used in software packages, such as the computer algebra system Wolfram Mathematica. For instance, in Mathematica, the command

`{1,0,Pi,E,I}[[3]]` returns `Pi`.

Here, the double square brackets `[[ ]]` enclose the independent variable, instead of the usual parentheses, reinforcing the idea of indexing as function evaluation. $\lhd$

Example 3.5. This is another special case of Example 3.3. Let $m, n \in \mathbb{N}$ and

$$D = \{(s,t) \in \mathbb{N} \times \mathbb{N} : s \leq m \wedge t \leq n\};$$

that is $D = \{1, \ldots, m\} \times \{1, \ldots, n\}$. Then $\mathbb{F}^D$ can be identified with the space $\mathbb{F}^{m \times n}$ of all $m \times n$ matrices with entries in $\mathbb{F}$. $\triangleleft$

Example 3.6. By $\mathbb{F}[z]$ we denote the set of all polynomials in variable z with coefficients from $\mathbb{F}$. Then $\mathbb{F}[z]$ is a vector space with addition and scalar multiplication defined pointwise. $\triangleleft$

Example 3.7. This is another special case of Example 3.3. Let $D = \mathbb{N}$. Then $\mathbb{F}^{\mathbb{N}}$ is the vector space of all sequences with entries in $\mathbb{F}$: the space of all real sequences if $\mathbb{F} = \mathbb{R}$, and the space of all complex sequences if $\mathbb{F} = \mathbb{C}$.

Prominent sequences in these spaces are those whose n-th entry (for a given $n \in \mathbb{N}$) is 1 and whose other entries are 0. $\triangleleft$

The next example is a generalization of Example 3.3,

Example 3.8. Let D be an arbitrary nonempty set and let $\mathscr{V}$ be a vector space over $\mathbb{F}$. Let $\mathscr{W}$ be the set of all functions from D to $\mathscr{V}$; that is $\mathscr{W} = \mathscr{V}^D$. With the addition and scaling of functions defined pointwise, $\mathscr{W}$ is a vector space over $\mathbb{F}$. The functions in $\mathscr{V}^D$ are said to be *vector valued* functions. $\triangleleft$

4. Set operations in a vector space

In a set theory class, we learned about set operations. For two sets A and B, we defined $A \cap B$, $A \cup B$, $A \setminus B$, and $A \Delta B$. In a vector space $\mathscr{V}$ over $\mathbb{F}$, the exploration of subsets is further enriched by two additional operations: the addition of sets and the scaling of sets.

Definition 4.1. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\mathscr{A}$ and $\mathscr{B}$ be nonempty subsets of $\mathscr{V}$. We define the sum of $\mathscr{A} + \mathscr{B}$ by

$$\mathscr{A} + \mathscr{B} := \{u + v : u \in \mathscr{A}, v \in \mathscr{B}\}.$$

For $\alpha \in \mathbb{F}$ we define $\alpha\mathscr{A}$ by

$$\alpha\mathscr{A} := \{\alpha u : u \in \mathscr{A}\}.$$

Let $n \in \mathbb{N}$ and let $\mathscr{A}_1, \ldots, \mathscr{A}_n$ be subsets of $\mathscr{V}$. By recursion we define

$$\mathscr{A}_1 + \cdots + \mathscr{A}_k := (\mathscr{A}_1 + \cdots + \mathscr{A}_{k-1}) + \mathscr{A}_k, \quad k = 2, \ldots, n.$$

By Axiom **AA**, the set $\mathscr{A}_1 + \cdots + \mathscr{A}_n$ consists of all the sums $v_1 + \cdots + v_n$ where $v_j \in \mathscr{A}_j$ for all $j \in \{1, \ldots, n\}$. $\triangleleft$

5. Special subsets of a vector space

The following definition distinguishes important subsets of a vector space $\mathscr{V}$ over $\mathbb{F}$.

Definition 5.1. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$. A subset $\mathscr{U}$ of $\mathscr{V}$ is said to be a *subspace* of $\mathscr{V}$ if the following three conditions are satisfied:

SuZ. $0_{\mathscr{V}} \in \mathscr{U}$.
SuA. For every $u, v \in \mathscr{U}$ we have $u + v \in \mathscr{U}$.
SuS. For every $\alpha \in \mathbb{F}$ and every $u \in \mathscr{U}$ we have $\alpha u \in \mathscr{U}$. ◁

The property **SuA** in Definition 5.1 is stated in words as: $\mathscr{U}$ is closed under addition. Using set addition this property can be written as $\mathscr{U} + \mathscr{U} \subseteq \mathscr{U}$. The property **SuS** is stated in words as: $\mathscr{U}$ is closed under scalar multiplication. In terms of set operations, this can be written as: $\alpha\mathscr{U} \subseteq \mathscr{U}$ for all $\alpha \in \mathbb{F}$.

Proposition 5.2. *An intersection of subspaces of a vector space is also a subspace.*

Proposition 5.3. *A sum of subspaces of a vector space is also a subspace.*

A union of subspaces of a vector space is not necessarily a vector space. Problems 7.6 and 7.8 deal with this question.

Definition 5.4. Let $\mathscr{A}$ be a nonempty subset of $\mathscr{V}$. The *span of* $\mathscr{A}$ is the set of all linear combinations of vectors in $\mathscr{A}$. The span of $\mathscr{A}$ is denoted by

$$\operatorname{span}(\mathscr{A}).$$

The span of the empty set is the trivial vector space $\{0_{\mathscr{V}}\}$; that is,

$$\operatorname{span}(\emptyset) = \{0_{\mathscr{V}}\}.$$

If

$$\mathscr{V} = \operatorname{span}(\mathscr{A}),$$

then $\mathscr{A}$ is said to be a *spanning set* for $\mathscr{V}$. ◁

It is useful to write the definition of a span in set-builder notation. Let $\mathscr{A}$ be a nonempty subset of $\mathscr{V}$. Then

$$\operatorname{span}(\mathscr{A}) = \left\{ v \in \mathscr{V} : \begin{array}{l} \exists m \in \mathbb{N}, \\ \exists \alpha_1, \dots, \alpha_m \in \mathbb{F}, \\ \exists u_1, \dots, u_m \in \mathscr{A}, \\ \text{such that } v = \displaystyle\sum_{k=1}^{m} \alpha_k u_k \end{array} \right\}.$$

This notation is somewhat heavy. In words: $\operatorname{span}(\mathscr{A})$ is the set of all vectors $v \in \mathscr{V}$ such that there exists $m \in \mathbb{N}$, there exist scalars $\alpha_1, \dots, \alpha_m$ in $\mathbb{F}$, and there exist vectors $u_1, \dots, u_m$ in $\mathscr{A}$ such that $v = \sum_{k=1}^{m} \alpha_k u_k$.

Theorem 5.5. *Let* $\mathscr{A} \subseteq \mathscr{V}$. *Then* $\operatorname{span}(\mathscr{A})$ *is a subspace of* $\mathscr{V}$.

Proof. If $\mathscr{A} = \emptyset$, then by the usual convention $\operatorname{span}(\emptyset) = \{0_{\mathscr{V}}\}$, which is a subspace of $\mathscr{V}$. So assume $\mathscr{A} \neq \emptyset$.

We verify the three defining properties of a subspace.

SuZ: $0_{\mathscr{V}} \in \operatorname{span}(\mathscr{A})$. Choose any $u_1 \in \mathscr{A}$ (possible since $\mathscr{A} \neq \varnothing$). Let $m = 1$ and $\alpha_1 = 0$. Then

$$\sum_{k=1}^{1} \alpha_k u_k = 0 \cdot u_1 = \mathbf{0},$$

so $\mathbf{0} \in \operatorname{span}(\mathscr{A})$ by the definition of span.

SuA: $\operatorname{span}(\mathscr{A})$ respects addition. Let $v, w \in \operatorname{span}(\mathscr{A})$. By definition of span, there exist $m, n \in \mathbb{N}$, scalars $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ and $\beta_1, \ldots, \beta_n \in \mathbb{F}$, and vectors $u_1, \ldots, u_m \in \mathscr{A}$ and $v_1, \ldots, v_n \in \mathscr{A}$ such that

$$u = \sum_{k=1}^{m} \alpha_k u_k, \qquad v = \sum_{j=1}^{n} \beta_j v_j.$$

Then

$$u + v = \sum_{k=1}^{m} \alpha_k u_k + \sum_{j=1}^{n} \beta_j v_j.$$

This is again a linear combination of vectors from $\mathscr{A}$: indeed, letting $p = m + n$, define

$$w_l = \begin{cases} u_l, & l \in \{1, \ldots, m\}, \\ v_{l-m}, & l \in \{m+1, \ldots, p\}, \end{cases} \qquad \gamma_l = \begin{cases} \alpha_l, & l \in \{1, \ldots, m\}, \\ \beta_{l-m}, & l \in \{m+1, \ldots, p\}. \end{cases}$$

Then $w_1, \ldots, w_p \in \mathscr{A}$, $\gamma_1, \ldots, \gamma_p \in \mathbb{F}$, and

$$u + v = \sum_{l=1}^{p} \gamma_l w_l.$$

Hence $v + w \in \operatorname{span}(\mathscr{A})$.

SuS: $\operatorname{span}(\mathscr{A})$ respects scalar multiplication. Let $v \in \operatorname{span}(\mathscr{A})$ and let $\lambda \in \mathbb{F}$. Choose $m \in \mathbb{N}$, scalars $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$, and vectors $v_1, \ldots, v_m \in \mathscr{A}$ with

$$v = \sum_{k=1}^{m} \alpha_k v_k.$$

Then

$$\lambda v = \lambda \sum_{k=1}^{m} \alpha_k v_k = \sum_{k=1}^{m} (\lambda \alpha_k) v_k,$$

which is a linear combination of elements of $\mathscr{A}$. Thus $\lambda v \in \operatorname{span}(\mathscr{A})$.

Since $\operatorname{span}(\mathscr{A})$ contains $0_{\mathscr{V}}$ and is closed under vector addition and scalar multiplication, it is a subspace of $\mathscr{V}$. □

Proposition 5.6. *If $\mathscr{U}$ is a subspace of $\mathscr{V}$ and $\mathscr{A} \subseteq \mathscr{U}$, then* $\operatorname{span}(\mathscr{A}) \subseteq \mathscr{U}$*.*

Proof. Write a proof as an exercise. □

Definition 5.7. Let $\mathscr{V}$ be a vector space over $\mathbb{R}$. A nonempty subset $\mathscr{C}$ of $\mathscr{V}$ is said to be a *cone* in $\mathscr{V}$ if $\alpha\mathscr{C} \subseteq \mathscr{C}$ for all $\alpha > 0$. ◁

Definition 5.8. Let $\mathscr{V}$ be a vector space over $\mathbb{R}$. A nonempty subset $\mathscr{S}$ of $\mathscr{V}$ is said to be a *convex subset* of $\mathscr{V}$ if $\alpha u + (1-\alpha)v \in \mathscr{S}$ for all $\alpha \in [0,1]$ and all $u, v \in \mathscr{S}$. ◁

Exercise 5.9. Let $\mathscr{V}$ be a vector space over $\mathbb{R}$ and let $\mathscr{C}$ be a cone in $\mathscr{V}$. Prove that $\mathscr{C}$ is a convex set if and only if $\mathscr{C} + \mathscr{C} \subseteq \mathscr{C}$. ◁

6. Direct sums of subspaces

Let $\mathscr{V}$ be a vector space over $\mathbb{F}$. Let $\mathscr{X}$ and $\mathscr{Y}$ be subspaces of $\mathscr{V}$. Recall that $v \in \mathscr{X} + \mathscr{Y}$ if and only if there exist $x \in \mathscr{X}$ and $y \in \mathscr{Y}$ such that $v = x + y$. A stronger version of the last statement is in the following definition.

Definition 6.1. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\mathscr{X}$ and $\mathscr{Y}$ be subspaces of $\mathscr{V}$. The sum $\mathscr{X}+\mathscr{Y}$ is called a *direct sum* if for every $v \in \mathscr{X}+\mathscr{Y}$ there exist unique $x \in \mathscr{X}$ and unique $y \in \mathscr{Y}$ such that $v = x+y$. The direct sum is denoted by $\mathscr{X} \oplus \mathscr{Y}$. Formally, the sum $\mathscr{X} + \mathscr{Y}$ is direct if the following implication holds: for all $x_1, x_2 \in \mathscr{X}$ and for all $y_1, y_2 \in \mathscr{Y}$

$$x_1 + y_1 = x_2 + y_2 \quad \Rightarrow \quad x_1 = x_2 \ \wedge \ y_1 = y_2. \tag{6.1}$$

◁

Example 6.2. Let $\mathbb{F} = \mathbb{R}$, $\mathscr{V} = \mathbb{R}^4$,

$$\mathscr{X} = \bigl\{(x_1, x_2, x_3, 0) : x_1, x_2, x_3 \in \mathbb{R}\bigr\}$$

and

$$\mathscr{Y} = \bigl\{(0, y_1, y_2, y_3) : y_1, y_2, y_3 \in \mathbb{R}\bigr\}.$$

Then $\mathbb{R}^4 = \mathscr{X} + \mathscr{Y}$. However, this sum is not a direct sum. For $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ we can take $\mathbf{x} = (v_1, s_2, s_3, 0) \in \mathscr{X}$ and $\mathbf{y} = (0, v_2 - s_2, v_3 - s_3, v_4) \in \mathscr{Y}$ with $s_2, s_3 \in \mathbb{R}$ arbitrary.

Setting

$$\mathscr{X} = \bigl\{(x_1, x_2, x_2, 0) : x_1, x_2 \in \mathbb{R}\bigr\} \quad \text{and} \quad \mathscr{Y} = \bigl\{(0, -y_1, y_1, y_2) : y_1, y_2 \in \mathbb{R}\bigr\},$$

we have $\mathbb{R}^4 = \mathscr{X} \oplus \mathscr{Y}$. Prove this as an exercise. ◁

Proposition 6.3. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\mathscr{X}$ and $\mathscr{Y}$ be subspaces of $\mathscr{V}$. The following statements are equivalent:*

(a) *The sum $\mathscr{X} + \mathscr{Y}$ is direct.*
(b) *For all $x, y \in \mathscr{V}$ we have*

$$x \in \mathscr{X} \ \wedge \ y \in \mathscr{Y} \ \wedge \ x + y = 0_{\mathscr{V}} \quad \Rightarrow \quad x = y = 0_{\mathscr{V}}. \tag{6.2}$$

(c) $\mathscr{X} \cap \mathscr{Y} = \{0_{\mathscr{V}}\}$.

Proof. The implication in (6.2) is a special case of the implication in (6.1). Let $x \in \mathscr{X}$, let $y \in \mathscr{Y}$, and assume $x+y = 0_{\mathscr{V}}$. Then we have $x+y = 0_{\mathscr{V}}+0_{\mathscr{V}}$, and since $0_{\mathscr{V}} \in \mathscr{X}$ and $0_{\mathscr{V}} \in \mathscr{Y}$, the implication in (6.1) yields $x = 0_{\mathscr{V}}$ and $y = 0_{\mathscr{V}}$. This proves (a) implies (b).

Assume (b). Let $v \in \mathscr{X} \cap \mathscr{Y}$ be arbitrary. Since $\mathscr{X} \cap \mathscr{Y}$ is a subspace, $-v \in \mathscr{X} \cap \mathscr{Y}$. Set $x = v$, $y = -v$ in (b). Then (b) implies $v = 0_{\mathscr{V}}$. This proves (c).

Assume (c). We need to prove the implication in (6.1). Let $x_1, x_2 \in \mathscr{X}$ and $y_1, y_2 \in \mathscr{Y}$ be arbitrary and assume that $x_1 + y_1 = x_2 + y_2$. Then by algebra in $\mathscr{V}$ we have

$$0_{\mathscr{V}} = (x_1 + y_1) - (x_2 + y_2) = (x_1 - x_2) - (y_2 - y_1).$$

Consequently,

$$x_1 - x_2 = y_2 - y_1.$$

Since $\mathscr{X}$ is a subspace, $x_1 - x_2 \in \mathscr{X}$ and since $\mathscr{Y}$ is a subspace, $y_2 - y_1 \in \mathscr{Y}$. Therefore,

$$x_1 - x_2 = y_2 - y_1 \in \mathscr{X} \cap \mathscr{Y} = \{0_{\mathscr{V}}\}.$$

Consequently, $x_1 = x_2$ and $y_1 = y_2$. This proves the implication in (6.1), proving (a).

Since we proved (a)⇒(b)⇒(c)⇒(a), the propositions is proved. □

Definition 6.4. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$, let $n \in \mathbb{N}$ and let $\mathscr{X}_1, \ldots, \mathscr{X}_n$ be subspaces of $\mathscr{V}$. The sum $\mathscr{X}_1 + \cdots + \mathscr{X}_n$ is called a *direct sum* if for every $v \in \mathscr{X}_1 + \cdots + \mathscr{X}_n$ there exist unique $x_j \in \mathscr{X}_j$, $j \in \{1, \ldots, n\}$, such that $v = x_1 + \cdots + x_n$. The direct sum is denoted by $\mathscr{X}_1 \oplus \cdots \oplus \mathscr{X}_n$. ◁

The preceding definition of the direct sum of subspaces written as an implication is as follows: For all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathscr{V}$ the following implication holds

$$\forall k \in \{1, \ldots, n\}\ x_k, y_k \in \mathscr{X}_k \quad \wedge \quad \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k \\ \Rightarrow \quad \forall k \in \{1, \ldots, n\} \ \ x_k = y_k. \quad (6.3)$$

Proposition 6.5. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$, let $n \in \mathbb{N}$ and let $\mathscr{X}_1, \ldots, \mathscr{X}_n$ be subspaces of $\mathscr{V}$. The following statements are equivalent:*

(a) *The sum $\mathscr{X}_1 + \cdots + \mathscr{X}_n$ is direct.*

(b) *For all $x_1, \ldots, x_n \in \mathscr{V}$ the following implication holds*

$$0_{\mathscr{V}} = \sum_{k=1}^{n} x_k \ \wedge \ \forall k \in \{1, \ldots, n\} \ \ x_k \in \mathscr{X}_k \\ \Rightarrow \quad \forall k \in \{1, \ldots, n\}\ x_k = 0_{\mathscr{V}}. \quad (6.4)$$

Proof. Assume (a). That is, assume that the implication in (6.3) holds. Setting $v = 0_{\mathscr{V}}$ and $y_k = 0_{\mathscr{V}}$ for all $k \in \{1, \ldots, n\}$ in (6.3), the implication in (6.3) becomes (6.4). This proves (a)⇒(b).

Assume (b). To prove the implication in (6.3), let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathscr{V}$ be arbitrary and assume

$$\forall k \in \{1, \ldots, n\} \;\; x_k, y_k \in \mathscr{X}_k \quad \wedge \quad \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k.$$

The preceding assumption yields

$$0_{\mathscr{V}} = \sum_{k=1}^{n} (x_k - y_k) \;\wedge\; \forall k \in \{1, \ldots, n\} \;\; x_k - y_k \in \mathscr{X}_k.$$

Now, by (6.4) we deduce

$$\forall k \in \{1, \ldots, n\} \quad x_k - y_k = 0_{\mathscr{V}}.$$

This proves the implication in (6.3), proving (b).

The proposition is proved. □

In the next theorem we prove that the Cartesian product of two vector spaces with appropriately defined vector addition and scalar multiplication is a vector space.

Theorem 6.6. *Let $\mathscr{V}$ and $\mathscr{X}$ be a vector spaces over $\mathbb{F}$. Define the vector addition and scalar multiplication on the Cartesian product $\mathscr{V} \times \mathscr{X}$ as follows. For all $v, w \in \mathscr{V}$, all $x, y \in \mathscr{X}$ and all $\alpha \in \mathbb{F}$ set*

$$(v, x) + (w, y) = (v + w, x + y), \qquad \alpha(v, x) = (\alpha v, \alpha x). \tag{6.5}$$

The set $\mathscr{V} \times \mathscr{X}$ with these two operations is a vector space.

Remark 6.7. Notice that the first plus sign in (6.5) is the addition in $\mathscr{V} \times \mathscr{X}$ which is being defined, the second plus sign is the addition in $\mathscr{V}$ and the third plus sign is the addition in $\mathscr{X}$. ◁

Definition 6.8. The set $\mathscr{V} \times \mathscr{X}$ with the operations defined in (6.5) is called the *direct product* of the vector spaces $\mathscr{V}$ and $\mathscr{X}$. ◁

7. Problems

Problem 7.1. In Definition 1.2 we use the same symbol + to denote to different additions; one addition is the addition of complex numbers in $\mathbb{F}$, the other addition is the addition of vectors in the vector space $\mathscr{V}$. Similarly, the usage of the blank space between two symbols is ambiguous; between two complex numbers it means the product of two complex numbers, while between a complex number and a vector in $\mathscr{V}$ means scaling of that vector by a complex number. As a learner you should pay attention and make sure that you understand the meaning of the formulas that you are dealing with.

Let us introduce some "funny" names for the algebraic operations that appear in Definition 1.2.

$$\mathsf{VectorPlus} : \mathscr{V} \times \mathscr{V} \to \mathscr{V}, \qquad \mathsf{Scale} : \mathbb{F} \times \mathscr{V} \to \mathscr{V}$$
$$\mathsf{Plus} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}, \qquad \mathsf{Times} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}.$$

Thus, for $u, v \in \mathscr{V}$ the sum of the vectors u and v is denoted by $\mathsf{VectorPlus}(u, v)$, for $\alpha \in \mathbb{F}$ and $v \in \mathscr{V}$ the scaling of the vector v by α is denoted by $\mathsf{Scale}(\alpha, v)$, for $\alpha, \beta \in \mathbb{F}$ the sum of the complex numbers α and β is denoted by $\mathsf{Plus}(\alpha, \beta)$, and for $\alpha, \beta \in \mathbb{F}$ the product of the complex numbers α and β is denoted by $\mathsf{Times}(\alpha, \beta)$.

Just to clarify, in this notation we have $\mathsf{Plus}(2, 3) = 5$ and $\mathsf{Times}(2, 3) = 6$. The distributive law for complex numbers in this notation reads: for all complex numbers α, β and γ we have

$$\mathsf{Times}\big(\alpha, \mathsf{Plus}(\beta, \gamma)\big) = \mathsf{Plus}\big(\mathsf{Times}(\alpha, \beta), \mathsf{Times}(\alpha, \gamma)\big).$$

Finally, your task in this problem is to rewrite the axioms **SA**, **SD**, **SD**, and **SO** using the notation for the algebraic operations introduced above. $\lhd$

Problem 7.2. Let $\mathbb{R}_+$ denote the set of positive real numbers, set

$$\mathscr{V} = \mathbb{R}_+ \times \mathbb{R}_+ = (\mathbb{R}_+)^2,$$

and let $\mathbb{F} = \mathbb{R}$. Define the addition and the scalar multiplication in $\mathscr{V}$ as follows: For all $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathscr{V}$ and all $\alpha \in \mathbb{F}$ set

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 v_1 \\ u_2 v_2 \end{bmatrix}, \qquad \alpha \odot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} (v_1)^\alpha \\ (v_2)^\alpha \end{bmatrix}.$$

Prove that $\mathscr{V}$ with the vector addition $\oplus$ and the vector scaling $\odot$ is a vector space over $\mathbb{R}$. $\lhd$

Problem 7.3. In this problem $(-1, 1)$ denotes the open interval of real numbers. That is,

$$(-1, 1) = \big\{x \in \mathbb{R} : -1 < x \ \wedge \ x < 1\big\}.$$

For $u, v, x, y, z \in \mathbb{R}$ with $x > 0$ and $z \neq 0$ by

$$u + v, \quad u - v, \quad uv, \quad \frac{y}{z}, \quad x^u$$

we denote the standard algebraic operations in $\mathbb{R}$. Set $\mathscr{V} = (-1, 1)$ and let $\mathbb{F} = \mathbb{R}$. Define the vector addition and the scalar multiplication

$$\oplus : \mathscr{V} \times \mathscr{V} \to \mathscr{V}, \qquad \odot : \mathbb{R} \times \mathscr{V} \to \mathscr{V}$$

on $\mathscr{V}$ by: For all $u, v \in \mathscr{V}$ and all $\alpha \in \mathbb{R}$ set

$$u \oplus v = \frac{u + v}{1 + uv}, \qquad \alpha \odot v = \frac{(1 + v)^\alpha - (1 - v)^\alpha}{(1 + v)^\alpha + (1 - v)^\alpha}.$$

Prove that $\mathscr{V}$ with the vector addition $\oplus$ and the scaling $\odot$ is a vector space over $\mathbb{R}$. $\lhd$

Problem 7.4. Consider the vector space $\mathbb{R}^\mathbb{R}$ of all real valued functions defined on $\mathbb{R}$. This vector space is considered over the field $\mathbb{R}$. The purpose of this exercise is to study some special subspaces of the vector space $\mathbb{R}^{\mathbb{R}}$. Let $\omega \in \mathbb{R}$ be arbitrary. Consider the set

$$\mathscr{S}_\omega := \Bigl\{ f \in \mathbb{R}^{\mathbb{R}} : \exists a, b \in \mathbb{R} \ \text{ such that } \ f(t) = a \sin(\omega t + b) \ \ \forall t \in \mathbb{R} \Bigr\}.$$

(a) Do you see exceptional values for ω for which the set $\mathscr{S}_\omega$ is particularly simple? State them and explain why they are special. Here I used plural just in case that there are several special cases. However, it is possible that there is only one special case for ω.
(b) Prove that $\mathscr{S}_\omega$ is a subspace of $\mathbb{R}^{\mathbb{R}}$. (Hint: Except for the special case, this problem should be solved by writing the set $\mathscr{S}_\omega$ as a span of two linearly independent famous functions. One should use only basic trigonometry and polar coordinates.)
(c) For each $\omega \in \mathbb{R}$ find a basis for $\mathscr{S}_\omega$. Plot the function $\omega \mapsto \dim \mathscr{S}_\omega$ with $\omega \in \mathbb{R}$.
(d) For all $\psi, \omega \in \mathbb{R}$ calculate $\dim(\mathscr{S}_\psi \cap \mathscr{S}_\omega)$.
(e) Find all pairs $(\psi, \omega) \in \mathbb{R}^2$ for which $\mathscr{S}_\psi \cup \mathscr{S}_\omega$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.
(f) For all $\psi, \omega \in \mathbb{R}$ calculate $\dim\bigl(\mathscr{S}_\psi + \mathscr{S}_\omega\bigr)$. $\triangleleft$

Some items in this problem belong to the next section of the notes.

Problem 7.5. Even and Odd Functions. Let D be a nonempty set. Let $\mathbb{F}^D$ be a vector space of all functions with the domain D and codomain $\mathbb{F}$. Denote by $\mathscr{D} \subseteq \mathbb{F}^D$ the set of all constant functions in $\mathbb{F}^D$. That is

$$\mathscr{D} = \bigl\{ f \in \mathbb{F}^D : \exists c \in \mathbb{F} \ \text{ such that } \ \forall t \in D \ \ f(t) = c \bigr\}.$$

Let $\varphi : D \to D$ be a bijection. Set

$$\mathscr{E}_\varphi = \bigl\{ f \in \mathbb{F}^D : f\bigl(\varphi(t)\bigr) = f(t) \ \forall t \in D \bigr\},$$
$$\mathscr{O}_\varphi = \bigl\{ f \in \mathbb{F}^D : f\bigl(\varphi(t)\bigr) = -f(t) \ \forall t \in D \bigr\}.$$

(a) Prove that $\mathscr{E}_\varphi$ and $\mathscr{O}_\varphi$ are subspaces of $\mathbb{F}^D$.
(b) Prove $\mathscr{E}_\varphi \cap \mathscr{O}_\varphi = \bigl\{ 0_{\mathbb{F}^D} \bigr\}$.
(c) Prove that for an arbitrary bijection $\varphi : D \to D$ we have $\mathscr{D} \subseteq \mathscr{E}_\varphi$.
(d) In this item we explore the extreme cases for $\mathscr{E}_\varphi$ and $\mathscr{O}_\varphi$.
  (i) Characterize the bijections $\varphi : D \to D$ such that $\mathscr{E}_\varphi = \mathscr{D}$. (For all $s, t \in D$ there exists $k \in \mathbb{N}$ such that $\varphi^k(s) = t$ or $\varphi^k(t) = s$, that is the entire D is one cycle.)
  (ii) Characterize the bijections $\varphi : D \to D$ such that $\mathscr{E}_\varphi = \mathbb{F}^D$.
  (iii) Characterize the bijections $\varphi : D \to D$ such that $\mathscr{O}_\varphi = \bigl\{ 0_{\mathbb{F}^D} \bigr\}$.
  (iv) Characterize the bijections $\varphi : D \to D$ such that $\mathscr{O}_\varphi = \mathbb{F}^D$.
(e) Explore if there is a relationship between the following three pairs of subspaces
  (i) $\mathscr{E}_\varphi$, $\mathscr{O}_\varphi$.
  (ii) $\mathscr{E}_{\varphi^{-1}}$, $\mathscr{O}_{\varphi^{-1}}$.
  (Here $\varphi^{-1} : D \to D$ is the inverse bijection of $\varphi : D \to D$.)
  (iii) $\mathscr{E}_{\varphi \circ \varphi}$, $\mathscr{O}_{\varphi \circ \varphi}$.

(f) Using the definitions and notation introduced earlier in this problem, characterize all functions in the set $\mathscr{E}_\varphi \oplus \mathscr{O}_\varphi$. In particular, show that $\mathscr{E}_\varphi \oplus \mathscr{O}_\varphi$ can be written in a simple closed form using the notation established in this problem.

(g) Determine a necessary and sufficient condition on the bijection $\varphi : D \to D$ under which the decomposition $\mathbb{R}^D = \mathscr{E}_\varphi \oplus \mathscr{O}_\varphi$ holds.

Notes:

(1) Parts of this problem are challenging. Exploring examples can guide your thinking. Create your own examples and think through the suggested examples below.

(2) Let D be an arbitrary nonempty set and let $\iota(t) = t$ be the identity bijection on D. Describe $\mathscr{E}_\iota$ and $\mathscr{O}_\iota$ and think through the rest of the problem in this trivial case.

(3) This problem is inspired by the concepts of odd and even functions, which are first encountered in a precalculus class. In the precalculus setting we have $D = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$ and $\varphi(t) = -t$ for all $t \in \mathbb{R}$. To build intuition, it is instructive to first consider the given problem in this familiar precalculus framework. In this setting, the prominent examples of even and odd functions are many. The most notable ones are the hyperbolic cosine and hyperbolic sine:

$$\cosh(t) = \frac{1}{2}\big(\exp(t) + \exp(-t)\big), \quad \forall t \in \mathbb{R},$$

$$\sinh(t) = \frac{1}{2}\big(\exp(t) - \exp(-t)\big), \quad \forall t \in \mathbb{R}.$$

Verify that cosh is an even function and sinh is an odd function. Furthermore, verify and internalize the fundamental identity

$$e^t = \exp(t) = \cosh(t) + \sinh(t), \quad \forall t \in \mathbb{R}.$$

(4) Let $a \in \mathbb{R}$. Consider $D = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$ and $\varphi(t) = t + a$. Describe $\mathscr{E}_\varphi$ and $\mathscr{O}_\varphi$. What is $\varphi \circ \varphi$? What is the relationship between $\mathscr{E}_\varphi$, $\mathscr{O}_\varphi$, $\mathscr{E}_{\varphi\circ\varphi}$, and $\mathscr{O}_{\varphi\circ\varphi}$?

(5) Let $D = \{1, 2, 3\}$, $\mathbb{F} = \mathbb{R}$. There are six bijections on D:

t	$\varphi_1(t)$	$\varphi_2(t)$	$\varphi_3(t)$	$\varphi_4(t)$	$\varphi_5(t)$	$\varphi_6(t)$
1	1	1	2	2	3	3
2	2	3	1	3	1	2
3	3	2	3	1	2	1

Choose, one, two, or more of these bijections and explore questions asked in the problem for those bijections.

(6) Let $D = \{1, 2, 3, 4\}$, $\mathbb{F} = \mathbb{R}$. There are twenty-four bijections on D. In the table below, for $n \in \{1, \ldots, 24\}$ instead of $\varphi_n(t)$ we simply write n.

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	4	4	4	4	4	4
2	2	2	3	3	4	4	1	1	3	3	4	4	1	1	2	2	4	4	1	1	2	2	3	3
3	3	4	2	4	2	3	3	4	1	4	1	3	2	4	1	4	1	2	2	3	1	3	1	2
4	4	3	4	2	3	2	4	3	4	1	3	1	4	2	4	1	2	1	3	2	3	1	2	1

Choose, one, two, or more of these bijections and explore questions asked in the problem for those bijections.

(7) Let $n \in \mathbb{N} \setminus \{1\}$. Let $D = \{1, \dots, n\} \times \{1, \dots, n\}$, $\mathbb{F} = \mathbb{R}$. With this special choice of D, the vector space $\mathbb{R}^D$ can be identified with the vector space $\mathbb{R}^{n \times n}$ of all real $n \times n$ matrices. With this choice of D, discover a bijection $\varphi : D \to D$ such that $\mathscr{E}_\varphi$ is exactly the set of all symmetric $n \times n$ matrices. For the bijection φ that you discovered, characterize the matrices in $\mathscr{O}_\varphi$. Explore online whether the matrices in $\mathscr{O}_\varphi$ have a common name associated with them.

$\triangleleft$

Problem 7.6. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$. Let $\mathscr{U}$ and $\mathscr{W}$ be subspaces of $\mathscr{V}$. Prove that $\mathscr{U} \cup \mathscr{W}$ is a subspace of $\mathscr{V}$ if and only if $\mathscr{U} \subseteq \mathscr{W}$ or $\mathscr{W} \subseteq \mathscr{U}$. $\triangleleft$

Problem 7.7. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $n \in \mathbb{N}$, $n > 2$. Let $\mathscr{U}_1, \dots, \mathscr{U}_n$ be subspaces of $\mathscr{V}$. If the union $\mathscr{U}_1 \cup \cdots \cup \mathscr{U}_n$ is a subspace, then

$$\mathscr{U}_1 \subseteq \mathscr{U}_2 \cup \cdots \cup \mathscr{U}_n \quad \text{or} \quad \mathscr{U}_n \subseteq \mathscr{U}_1 \cup \cdots \cup \mathscr{U}_{n-1}. \tag{7.1}$$

$\triangleleft$

Proof. We will prove the contrapositive. Assume that (7.1) is not true. Then there exist $u_1 \in \mathscr{U}_1$ such that $u_1 \notin \mathscr{U}_j$ for all $j \in \{2, \dots, n\}$ and there exist $u_n \in \mathscr{U}_n$ such that $u_n \notin \mathscr{U}_j$ for all $j \in \{1, \dots, n-1\}$.

Let $\alpha \in \mathbb{F} \setminus \{0\}$. Then $\alpha u_n \in \mathscr{U}_n$ since $\mathscr{U}_n$ is a subspace and, since $\alpha \neq 0$, $\alpha u_n \notin \mathscr{U}_j$ for all $j \in \{1, \dots, n-1\}$.

Since $u_1 \in \mathscr{U}_1$ and $\alpha u_n \notin \mathscr{U}_1$ we have $u_1 + \alpha u_n \notin \mathscr{U}_1$ for all $\alpha \in \mathbb{F} \setminus \{0\}$.

Since $u_1 \notin \mathscr{U}_n$ and $\alpha u_n \in \mathscr{U}_n$ we have $u_1 + \alpha u_n \notin \mathscr{U}_n$ for all $\alpha \in \mathbb{F}$.

Let $m \in \mathbb{N}$ be such that $1 < m < n$. (Since $n > 2$ such m exists.) By the choice of u_1 and u_n we have $u_1 \notin \mathscr{U}_m$ and $\alpha u_n \notin \mathscr{U}_m$ for all $\alpha \in \mathbb{F} \setminus \{0\}$. Therefore, for at most one $\alpha \in \mathbb{F} \setminus \{0\}$ we can have $u_1 + \alpha u_n \in \mathscr{U}_m$. (If $u_1 + \alpha u_n \in \mathscr{U}_m$ and $u_1 + \beta u_n \in \mathscr{U}_m$ with $\alpha - \beta \neq 0$, then $(u_1 + \alpha u_n) - (u_1 + \beta u_n) = (\alpha - \beta) u_n \in \mathscr{U}_m$ with $\alpha - \beta \neq 0$ and $u_n \notin \mathscr{U}_m$ which is a contradiction.)

Thus, for at most $n - 2$ numbers $\alpha \in \mathbb{F} \setminus \{0\}$ we have

$$u_1 + \alpha u_n \in \mathscr{U}_1 \cup \cdots \cup \mathscr{U}_n.$$

Since the set $\mathbb{F} \setminus \{0\}$ is infinite, there exists $\alpha \in \mathbb{F} \setminus \{0\}$ such that

$$u_1 + \alpha u_n \notin \mathscr{U}_1 \cup \cdots \cup \mathscr{U}_n.$$

Recall that

$$u_1, u_n \in \mathscr{U}_1 \cup \cdots \cup \mathscr{U}_n.$$

The last two displayed relations show that $\mathscr{U}_1 \cup \cdots \cup \mathscr{U}_n$ is not a subspace of $\mathscr{V}$. □

Problem 7.8. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $n \in \mathbb{N}$. Let $\mathscr{U}_1, \ldots, \mathscr{U}_n$ be subspaces of $\mathscr{V}$. Prove that the union $\mathscr{U}_1 \cup \cdots \cup \mathscr{U}_n$ is a subspace if and only if there exists $m \in \{1, \ldots, n\}$ such that $\mathscr{U}_k \subseteq \mathscr{U}_m$ for all $k \in \{1, \ldots, n\}$. ◁

Problem 7.9. [Samantha Smith] Let $\mathscr{V}$ be a vector space over $\mathbb{F}$. Let $\mathscr{P}(\mathscr{V})$ be the power set of $\mathscr{V}$, that is the set of all subsets of $\mathscr{V}$. Set $\mathscr{W} = \mathscr{P}(\mathscr{V}) \setminus \{\emptyset\}$. Let the addition and scaling in $\mathscr{W}$ be defined as in Section 4. Is $\mathscr{W}$ with these two operations a vector space over $\mathbb{F}$? ◁

Problem 7.10. Let $\mathscr{V}$ be a real vector space, that is a vector space over $\mathbb{R}$. Set

$$\mathscr{V}_{\mathbb{C}} = \mathscr{V} \times \mathscr{V}.$$

Define the vector addition in $\mathscr{V}_{\mathbb{C}}$ as follows: For all $(u_1, v_1), (u_2, v_2) \in \mathscr{V}_{\mathbb{C}}$ set

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2).$$

In $\mathscr{V}_{\mathbb{C}}$, define the vector scaling with complex numbers as follows: For all $(u, v) \in \mathscr{V}_{\mathbb{C}}$ and all $\alpha, \beta \in \mathbb{R}$ set

$$(\alpha + \mathrm{i}\beta)(u, v) = (\alpha u + \beta v, \alpha v + \beta u).$$

(a) Prove that $\mathscr{V}_{\mathbb{C}}$ with the vector addition and the vector scaling with complex numbers defined as above is a complex vector space.

(b) Prove that function

$$\mathfrak{I} : \mathscr{V} \to \mathscr{V}_{\mathbb{C}}$$

defined by

$$\forall v \in \mathscr{V} \quad \mathfrak{I}(v) = (v, 0_{\mathscr{V}})$$

is an injection which has the following property:

$$\forall \alpha, \beta \in \mathbb{R} \;\; \forall u, v \in \mathscr{V} \quad \mathfrak{I}(\alpha u + \beta v) = \alpha \mathfrak{I}(u) + \beta \mathfrak{I}(v).$$

The mapping $\mathfrak{I}$ is called the *natural embedding* of $\mathscr{V}$ into $\mathscr{V}_{\mathbb{C}}$.

(c) The range of $\mathfrak{I}$ is the following subset of $\mathscr{V}_{\mathbb{C}}$:

$$\big\{(v, 0_{\mathscr{V}}) \in \mathscr{V}_{\mathbb{C}} : v \in \mathscr{V}\big\} = \mathscr{V} \times \{0_{\mathscr{V}}\}.$$

Prove that the set $\mathscr{V} \times \{0_{\mathscr{V}}\}$ is not a subspace of $\mathscr{V}_{\mathbb{C}}$.

(d) Prove that for all $u, v \in \mathscr{V}$ we have

$$(u, v) = (u, 0_{\mathscr{V}}) + \mathrm{i}(v, 0_{\mathscr{V}}).$$ ◁

Remark 7.11. (i) The complex vector space $\mathscr{V}_{\mathbb{C}}$, defined in Problem 7.10, is called the *complexification* of the real vector space $\mathscr{V}$.

(ii) Based on item (b) in Problem 7.10, it is common to identify the subset

$$\big\{(v, 0_{\mathscr{V}}) \in \mathscr{V}_{\mathbb{C}} : v \in \mathscr{V}\big\} = \mathscr{V} \times \{0_{\mathscr{V}}\}$$

with the set $\mathscr{V}$. With this identification, based on item (d) in Problem 7.10 we can write

$$\mathscr{V}_{\mathbb{C}} = \mathscr{V} + \mathrm{i}\mathscr{V}.$$

(iii) Let $n \in \mathbb{N}$. Applying the definition of $\mathscr{V}_{\mathbb{C}}$ to the real vector space $\mathbb{R}^n$ and using the observation in the preceding item we obtain that $\mathscr{V}_{\mathbb{C}} = \mathbb{C}^n$; that is

$$(\mathbb{R}^n)_{\mathbb{C}} = \mathbb{C}^n;$$

or in words: The complexification of the real vector space $\mathbb{R}^n$ is the complex vector space $\mathbb{C}^n$.

The beauty of Problem 7.10 lies in its universality: any real vector space $\mathscr{V}$ is embedded into a naturally defined complex vector space $\mathscr{V}_{\mathbb{C}}$. This construction allows us to study real vector spaces using the powerful tools that we will develop for complex vector spaces. $\lhd$